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A Definition of the Gravitational Field Inside Matter

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Using the procedure for an analogous situation in classical electromagnetic theory, a definition is proposed in general relativity for the gravitational field in the interior of a continuous distribution of matter. The procedure consists of introducing a small fictitious cavity inside the matter and proceeding to the limit when the cavity becomes vanishingly small.

1. INTRODUCTION

Classically, a "*field*" is a derived concept introduced to explain the primary experimental result that bits of the physical world affect each other in various ways. A system is analyzed into particles which act as sources for fields, which in turn act on other particles. The fields act on the other particles according to the *equations of motion*, and are related to their sources by the field equations. The field at a point is determined by the behavior of test particles (through the equations of motion).

However, in view of Einstein's principle of equivalence for the gravitational field, one cannot determine the gravitational field through the motion of only one geodesically moving test particle—a free test particle reveals no acceleration in a freely falling "elevator." On the other hand, it is well known that the gravitational field reveals itself through the relative acceleration of neighboring free test particles.

The relative acceleration of two geodesically moving test particles is given by the equation of geodesic deviation (Synge and Schild, 1956)

$$\frac{\delta^2\eta^a}{\delta s^2} + R^a_{.bcd}u^b\eta^c u^d = 0 \tag{1.1}$$

where η^a is the infinitesimal vector connecting the world lines, u^a is the unit vector tangent to one of the world lines such that $u_a\eta^a=0$, and s is the proper time along that world line. Equation (1.1) shows that the relative acceleration is determined by the Riemann curvature tensor, R_{abcd}, which is thus given a physical meaning (Pirani, 1965). Hence we may regard the Riemann curvature tensor as the gravitational field.

Now, it is well known that the Riemann curvature tensor has the following irreducible decomposition:

$$R^a_{.bcd}=C^a_{.bcd}+E^a_{.bcd}+\tfrac{1}{6}Rg^a_{.bcd} \tag{1.2}$$

where $C^a_{.bcd}$ is the Weyl conformal curvature tensor,

$$E^a_{.bcd}=\tfrac{1}{2}(\delta^a_c R_{bd}-\delta^a_d R_{bc}+g_{bd}R^a_c-g_{bc}R^a_d) \tag{1.3}$$

$$g^a_{.bcd}=\delta^a_d g_{bc}-\delta^a_c g_{bd} \tag{1.4}$$

where R_{ab} is the Ricci tensor and R the curvature invariant. In view of Einstein's field equations, the Ricci terms in (1.2) may be regarded as the source terms (which vanish in the absence of matter). Accordingly, we adopt in this paper the viewpoint that *the Weyl tensor, $C^a_{.bcd}$, is indeed the genuine gravitational field.*

However, the equation of geodesic deviation in (1.1) refers to particles moving in free space [where the curvature tensor reduces to the Weyl tensor in view of (1.2) and the field equations $R_{ab}=0$]. Hence, having adopted the Weyl tensor $C^a_{.bcd}$ as the genuine gravitational field, the problem naturally arises as to what is the gravitational field inside a continuous distribution of matter. The main purpose of this paper is to propose a definition of the gravitational field inside matter. In this paper, an attempt is made to gain some insight into the problem by considering the gravitational field inside a weak distribution of matter. The exact theory is considered in a subsequent paper.

2. A WEAK DISTRIBUTION OF MATTER

If we assume that the gravitational field due to a macroscopic body (consisting of perfect fluid) is sufficiently weak to be treated as first-order deviations from a Minkowskian space-time manifold, we obtain, to the order $1/c^2$, the linear metric form

$$ds^2=(1-2\Phi/c^2)(dx^0)^2-(1+2\Phi/c^2)(dx^\alpha)^2 \tag{2.1}$$

where

$$(dx^\alpha)^2\equiv(dx^1)^2+(dx^2)^2+(dx^3)^2$$

and Φ is the Newtonian gravitational potential satisfying Poisson's equation inside the body, c is the speed of light in vacuo, and $2\Phi \ll c^2$. Using the metric in (2.1), we find that the components of the Riemann curvature tensor (at any point inside the body) are given by

$$c^2 R_{\alpha\beta\gamma\epsilon} = (\Phi,_{\alpha\gamma}\delta_{\beta\epsilon} + \Phi,_{\beta\epsilon}\delta_{\alpha\gamma} - \Phi,_{\beta\gamma}\delta_{\alpha\epsilon} - \Phi,_{\alpha\epsilon}\delta_{\beta\gamma})$$

$$c^2 R_{0\beta\gamma 0} = -\Phi,_{\beta\gamma} \tag{2.2}$$

(where Greek suffixes run from 1 to 3) while the components of the corresponding Weyl tensor are given by

$$C_{\alpha\beta\gamma\epsilon} = R_{\alpha\beta\gamma\epsilon} - (8\pi k\rho/3c^2)\delta_{\alpha[\gamma}\delta_{\epsilon]\beta}$$

$$C_{0\beta\gamma 0} = R_{0\beta\gamma 0} + (4\pi k\rho/3c^2)\delta_{\beta\gamma} \tag{2.3}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta, [ ] denotes antisymmetrization, ρ is the proper density of the body, and k is the gravitational constant. Equation (2.3) shows that already in the linear theory, we can discern the Riemann tensor from the Weyl tensor [in view of the terms in (2.3) involving the proper density, ρ, of the body].

It follows from (2.2) and (2.3) that the linear gravitational field is given in terms of the Newtonian field $N_{\alpha\beta}$, where

$$N_{\alpha\beta} \equiv \Phi,_{\alpha\beta} \tag{2.4}$$

3. DEFINITION OF THE FIELD INSIDE MATTER

Suppose we introduce at any interior point P of matter a small fictitious cavity of arbitrary shape, volume τ, and surface σ. Suppose an observer inside the cavity uses (local) orthogonal Cartesian coordinate axes defined by a triad of (spacelike) unit vectors

$$e_A^{\alpha}$$

where the capital Latin letter labels the triads ($A = 1,2,3$) and the Greek letter α is a three-dimensional vector index ($\alpha = 1,2,3$). By means of local experiments, such an observer can determine, for example, the coordinates of a neighboring particle relative to his own frame. In this way, he obtains for the physical components of the Newtonian field $N_{\alpha\beta}$ the nine invariants N_{AB} defined by

$$N_{AB} \equiv e_A^{\alpha} e_B^{\beta} N_{\alpha\beta} \tag{3.1}$$

Consider now the invariants

$$\Phi_B = e_B^\beta \Phi,_\beta \tag{3.2}$$

On using the integral solution for Φ, we can rewrite (3.2) as

$$\Phi_B = \int \rho e_B^\beta (1/r),_\beta dv' \tag{3.3}$$

where r is the (Euclidean) distance from P to any volume element dv' of the distribution and ρ is assumed constant for the distribution of matter outside the cavity. The expression ρe_B^β in the integrand in (3.3) could be referred to as the *intensity of gravitation* in anology with the "intensity of magnetization" or "polarization vector" in the case of a dielectric.

Now, the volume integral in (3.3) can be rewritten as

$$\Phi_B = \int \left(e_B^\beta \rho / r\right),_\beta dv' - \int \left[\left(e_B^\beta \rho\right),_\beta / r\right] dv' \tag{3.4}$$

on using the identity

$$F^\alpha (1/r),_\alpha \equiv (F^\alpha / r),\alpha - (F^\alpha,_\alpha)/r \tag{3.5}$$

Hence, on transforming the first integral on the right-hand side of (3.4) to a surface integral (by means of Gauss' theorem), we obtain

$$\Phi_B = \int_\Sigma (\rho/r) e_B^\beta n_\beta d\Sigma + \int_\sigma (\rho/r) e_B^\beta n_\beta d\sigma \tag{3.6}$$

where Σ denotes the bounding surface of the entire distribution of matter, the unit vector n_β is the outward-drawn normal to Σ and σ, and where the second integral on the right-hand side of (3.4) vanishes for constant ρ and e_B^β.

It follows also from (3.6) and (3.1) that the physical components of the gravitational field as measured by an observer inside the cavity are given by

$$N_{AB} = \int_\Sigma (\rho/r),_\alpha e_A^\alpha e_B^\beta n_\beta d\Sigma + \int_\sigma (\rho/r),_\alpha e_A^\alpha e_B^\beta n_\beta d\sigma \tag{3.7}$$

Accordingly, we propose a definition of the field at the interior point P as

$$\lim_{\tau \to 0} \left(e_A^\alpha e_B^\beta N_{\alpha\beta}\right) \tag{3.8}$$

the limit being evaluated for a cavity of vanishingly small volume τ.

In view of the second surface integral in (3.7) and the proposed definition in (3.8), it follows that the field at the interior point will depend on the shape of the cavity introduced. For example, if the cavity is a circular cylinder with axis in the direction of e_B^β, then over the curved surface of the cylinder we have

$$e_B^\beta n_\beta = 0 \tag{3.9}$$

so that the normal component of the "intensity of gravitation" vanishes over the curved surface. However, the magnitude of the normal component of the "intensity of gravitation" over the circular ends is ρ. Hence, on taking the field point P at the midpoint of the axis of the circular cylinder, we find that the magnitude of the field produced at P by the surface distributions over the circular ends is

$$4\pi\rho(1-\cos\theta) \tag{3.10}$$

where θ is the angle subtended at P by a radius of the cylinder.

It then follows from (3.7) that the total field at P due to the entire distribution of matter has the scalar components

$$\int_\Sigma (\rho/r),_\alpha e_A^\alpha e_B^\beta n_\beta d\Sigma + 4\pi\rho_{AB}(1-\cos\theta) \tag{3.11}$$

which tends to the limit

$$\int_\Sigma (\rho/r),_\alpha e_A^\alpha e_B^\beta n_\beta d\Sigma \tag{3.12}$$

as $\theta \to 0$. Hence the field will be that measured in a *small cylindrical pipe*. On the other hand, if we let $\theta \to \pi/2$, the second term in (3.11) survives and gives rise to a term which may be regarded as the analog of the term $4\pi\mathbf{I}$ which occurs in the usual expression for magnetic induction, namely, $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I}$.

The above results appear to justify the view that $N_{\alpha\beta}$ occurring in the "geodesic deviation equation"

$$\ddot{\eta}_\alpha + N_\beta^\alpha \eta^\beta = 0$$

[which is the analog of (1.1)], or R_{abcd} replaced by C_{abcd} in (1.1), is the measure of the gravitational field.

4. ANALOGY WITH MAXWELL THEORY

In this section, we give a simple analogy between the linear gravitational field discussed above and the Maxwell electromagnetic field. Let the indices $\underline{1},\underline{2},\underline{3},\underline{4},\underline{5},\underline{6}$ denote pairs of tensorial indices according to the scheme

$$\underline{1}\equiv 23, \qquad \underline{2}\equiv 31, \qquad \underline{3}\equiv 12, \qquad \underline{4}\equiv 10, \qquad \underline{5}\equiv 20, \qquad \underline{6}\equiv 30$$

If, as suggested by our discussion above for the linear theory, we regard the Weyl tensor C_{AB} $(A,B=\underline{1},\underline{2},\underline{3},\underline{4},\underline{5},\underline{6})$ as the analog of the electromagnetic field skew tensor $F_{\alpha\beta}$, then we find from the usual expressions for $(\mathbf{E},\mathbf{H})$ in terms of $F_{\alpha\beta}$ that the ordered triplet

$$\left(C_{\underline{14}},C_{\underline{24}},C_{\underline{34}}\right)$$

corresponds to the electric vector

$$(E_x,E_y,E_z)$$

while the ordered triplet

$$\left(C_{\underline{23}},C_{\underline{31}},C_{\underline{12}}\right)$$

corresponds to the magnetic vector

$$(H_x,H_y,H_z)$$

From (2.3), we obtain

$$C_{\underline{14}}=C_{\underline{24}}=C_{\underline{34}}=0 \tag{4.1}$$

which are therefore independent of the shape of the cavity (employed in our proposed definition above) in much the same way as their analog (E_x,E_y,E_z). On the other hand, using (2.3) we obtain

$$\left(C_{\underline{23}},C_{\underline{31}},C_{\underline{12}}\right)=-\frac{1}{c^2}(\Phi_{,23},\Phi_{,13},\Phi_{,12}) \tag{4.2}$$

Hence from our previous discussion of $N_{\alpha\beta}$, it follows that the ordered triplet on the left-hand side of (4.2) depends on the shape of the cavity in much the same way as their analog (H_x,H_y,H_z).

5. CONCLUSIONS

The definition proposed above for the gravitational field inside matter appears plausible in the linear theory. Moreover, the implications of the definition bear striking analogies with ordinary electromagnetic theory.

However, it remains to be seen whether or not the proposed definition can be carried over to the exact theory. An attempt in this direction is reported in a subsequent paper.

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